Approximate identities. Positive functionals. GNS construction

(EXERCISES FOR LECTURES 7–10)

3.1. Let X be a locally compact Hausdorff topological space.
(a) Construct an approximate identity in $C_0(X)$.
(b) Show that $C_0(X)$ has a sequential approximate identity if and only if X is σ-compact.

3.2. Let H be a Hilbert space.
(a) Construct an approximate identity in $\mathscr{K}(H)$.
(b) Show that $\mathscr{K}(H)$ has a sequential approximate identity if and only if H is separable.

3.3. Prove that every separable C^*-algebra has a sequential approximate identity.

3.4. Let A be a normed algebra with a bounded approximate identity. Suppose that (e_n) is a sequential approximate identity in A.
(a) Is (e_n) necessarily bounded?
(b) Is (e_n) necessarily bounded under the assumption that A is a Banach algebra?

3.5. Let A be a strictly nonunital C^*-algebra, and (e_λ) be an approximate identity in A. Show that for each $a \in A_+$ we have $\|a\| = \sup_\lambda \|ae_\lambda\| = \lim_\lambda \|ae_\lambda\|$ (in particular, the limit exists).

Recall (see the lectures) that each **closed** two-sided ideal of a C^*-algebra is selfadjoint.

3.6. Construct a nonselfadjoint ideal in $C(\bar{\mathbb{D}})$ (where $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leqslant 1\}$).

3.7. Show that each two-sided ideal in $\mathscr{B}(H)$ is selfadjoint.

3.8. Let H be an infinite-dimensional separable Hilbert space, $\{e_n : n \geqslant 0\}$ be an orthonormal basis of H, and S be the right shift operator on H (i.e., a bounded linear operator uniquely determined by $S(e_n) = e_{n+1}$ for all n). Recall that the *Toeplitz algebra* is the C^*-subalgebra $\mathcal{T}$ of $\mathscr{B}(H)$ generated by S.
(a) Identify H with the Hardy space H^2 by taking each e_n to the function $e_n(z) = z^n$ on $\mathbb{T}$. Prove that $\mathcal{T}$ is generated by the set of all Toeplitz operators on H^2 with continuous symbols.
(b) Prove that $\mathscr{K}(H) \subset \mathcal{T}$, and that $\mathscr{K}(H)$ is the smallest closed 2-sided ideal of $\mathcal{T}$ containing all commutators $[a, b]$ $(a, b \in \mathcal{T})$.
(c) Construct an isometric $*$-isomorphism $\mathcal{T}/\mathscr{K}(H) \cong C(\mathbb{T})$.
(d) Show that the extension $0 \to \mathscr{K}(H) \to \mathcal{T} \xrightarrow{p} C(\mathbb{T}) \to 0$ does not split (i.e., there is no $*$-homomorphism $j \colon C(\mathbb{T}) \to \mathcal{T}$ satisfying $pj = \mathbf{1}_{C(\mathbb{T})}$). (*Hint:* use the Fredholm index.)

3.9. Prove that a linear functional $f \colon M_n \to \mathbb{C}$ is a state if and only if there exists a positive $S \in M_n$ with $\mathrm{Tr}(S) = 1$ such that $f(T) = \mathrm{Tr}(ST)$ $(T \in M_n)$.

3.10. Let G be a discrete group. Define $f \colon \ell^1(G) \to \mathbb{C}$ by $f(a) = a(e)$ (where e is the identity of G). Prove that f is a state, and that f uniquely extends to a state on $C^*_r(G)$ (see Definition 2.1).

3.11. Let A be C^* algebra, and let $B \subset A$ be a closed $*$-subalgebra. Recall (see the lectures) that every positive functional g on B extends to a positive functional f on A such that $\|f\| = \|g\|$.
(a) Is f necessarily unique?
(b) Show that, if B is a two-sided ideal of A, then f is unique and is given by $f(a) = \lim g(ae_\lambda) = \lim g(e_\lambda a)$, where (e_λ) is an approximate identity in B.

3.12. Let A be a C^*-algebra, and let f be a positive functional on A. Show that the GNS representation π_f is indeed a $*$-representation of A.

3.13. Let f be a positive functional on a C^*-algebra A, and let $t > 0$. Are the GNS representations (H_f, π_f) and (H_{tf}, π_{tf}) isomorphic? If yes, then construct a unitary isomorphism explicitly.

3.14. Let A be a C^*-algebra, let f be a positive functional on A, and let $\pi_f \colon A \to \mathscr{B}(H_f)$ be the GNS representation associated to f. Denote by Λ_f the canonical map from A to H_f, $\Lambda_f(a) = a + N_f$. Describe the triple (H_f, π_f, Λ_f) explicitly in the following cases:
(a) $A = C_0(X)$, where X is a locally compact Hausdorff topological space, and $f(a) = \int_X a\, d\mu$, where μ is a finite positive Radon measure on X.
(b) $A = M_n$, $f(T) = \frac{1}{n}\operatorname{Tr}(T)$.
(c) $A = \mathscr{K}(H)$, where H is a Hilbert space, and $f(a) = \langle ah \,|\, h \rangle$, where $h \in H \setminus \{0\}$.
(d) $A = C_r^*(G)$, where G is a discrete group, and f is uniquely determined by $f(a) = a(e)$ for $a \in \ell^1(G)$ (see Exercise 3.10).

3.15. Let A be a C^*-algebra, and let f be a positive functional on A. Denote by f_+ the canonical positive extension of f to A_+ (i.e., $f_+(a + \lambda 1_+) = f(a) + \lambda \|f\|$ for all $a \in A$, $\lambda \in \mathbb{C}$). Let (e_λ) be an approximate identity in A such that $e_\lambda \geqslant 0$ and $\|e_\lambda\| \leqslant 1$ for all λ (we do not assume that (e_λ) is monotone!).
(a) Show that (e_λ^2) is also an approximate identity having the same properties (see above).
(b) Show that $\lim f(e_\lambda) = \|f\|$ (for a monotone (e_λ), this was proved at the lecture).
(c) Deduce from (a) and (b) that $\Lambda_f(e_\lambda) \to \Lambda_{f_+}(1_+)$.
(d) Deduce from (c) that there is a unitary isomorphism $u \colon H_f \to H_{f_+}$ uniquely determined by $\Lambda_f(a) \mapsto \Lambda_{f_+}(a)$ $(a \in A)$.
(e) Show that $x_f = u^{-1}(\Lambda_f(1_+))$ is a cyclic vector for the GNS representation π_f uniquely determined by $\pi_f(a)x_f = \Lambda_f(a)$ $(a \in A)$.

3.16. Given A and f as in Exercise 3.14 (**a - d**), describe the cyclic vector x_f (see Exercise 3.15 (e)) explicitly.

3.17. Let A and f be as in Exercise 3.14 (a). Describe all cyclic vectors $x \in H_f$ for π_f. Give a necessary and sufficient condition for a cyclic vector $x \in H_f$ to satisfy $\langle \pi_f(a)x \,|\, x \rangle = f(a)$ $(a \in A)$.

3.18. Let A be a C^*-algebra, let f be a positive functional on A, and let π_f denote the GNS representation of A associated to f.
(a) Show that $\operatorname{Ker} \pi_f$ is the largest two-sided ideal of A contained in $\operatorname{Ker} f$.
(b) We say that f is *faithful* if $f(a) = 0$, $a \geqslant 0$ implies that $a = 0$. Show that if f is faithful, then π_f is faithful (i.e., $\operatorname{Ker} \pi_f = 0$).
(c) Is the converse of (b) true?

3.19. Let A be a separable C^*-algebra. Prove that there exists a state f on A such that the associated GNS representation π_f is faithful.