C^*-algebras. Functional calculus. Positivity

(EXERCISES FOR LECTURES 4–7)

2.1. Show that $C^n[0,1]$ is a Banach $*$-algebra under the involution $f^*(t) = \overline{f(t)}$ $(t \in [0,1])$, but is not a C^*-algebra unless $n = 0$.

2.2. Show that $\mathscr{A}(\bar{\mathbb{D}})$ is a Banach $*$-algebra under the involution $f^*(z) = \overline{f(\bar{z})}$ $(z \in \bar{\mathbb{D}})$, but is not a C^*-algebra.

2.3. Let G be a discrete group. Show that $\ell^1(G)$ is a Banach $*$-algebra under the involution $f^*(x) = \overline{f(x^{-1})}$ $(x \in G)$, but is not a C^*-algebra unless $G = \{e\}$.

2.4. **(a)** Does there exist a norm and an involution on $C^1[a,b]$ making it into a C^*-algebra?
(b) Does there exist a norm and an involution on $\mathscr{A}(\bar{\mathbb{D}})$ making it into a C^*-algebra?
(c) Does there exist a norm and an involution on $\ell^1(\mathbb{Z})$ making it into a C^*-algebra?
Remark. In 2.4 (a,b,c), we do not assume that the new norm is equivalent to the original norm.

2.5. Let X be a locally compact Hausdorff topological space, and let X_+ denote the one-point compactification of X. For each $f \in C_0(X)$, define $f_+ \colon X_+ \to \mathbb{C}$ by $f_+(x) = f(x)$ for $x \in X$ and $f_+(\infty) = 0$. Prove that f_+ is continuous, and that the map $C_0(X)_+ \to C(X_+)$, $f + \lambda 1_+ \mapsto f_+ + \lambda$, is an isometric $*$-isomorphism. (Here we assume that $C_0(X)_+$ is equipped with the canonical C^*-norm extending the supremum norm on $C_0(X)$.)

2.6. Let A and B be C^*-algebras. Show that if B is commutative, then each homomorphism from A to B is a $*$-homomorphism. Does the above result hold without the commutativity assumption?

2.7. Let G be a discrete group. The *left regular representation* $\lambda \colon \ell^1(G) \to \mathscr{B}(\ell^2(G))$ is given by $\lambda(f)g = f * g$ $(f \in \ell^1(G),\ g \in \ell^2(G))$. Prove that λ is well defined (that is, $f * g$ is defined everywhere on G and belongs to $\ell^2(G)$, that $\lambda(f)$ is a bounded linear operator, and that λ is a $*$-homomorphism). Prove that λ is faithful.

Definition 2.1. The *reduced group C^*-algebra* of G is the C^*-subalgebra $C^*_r(G) = \overline{\operatorname{Im}\lambda} \subset \mathscr{B}(\ell^2(G))$.

2.8. Let G be a discrete abelian group. Construct an isometric $*$-isomorphism $C^*_r(G) \cong C(\widehat{G})$.

2.9. Let $A = C^1[0,1]$. Is it true that **(a)** for each unitary $u \in A$ we have $\sigma(u) \subset \mathbb{T}$? **(b)** for each selfadjoint $a \in A$ we have $\sigma(a) \subset \mathbb{R}$? **(c)** for each $a \in A$ we have $\|a\| = r(a)$?

2.10. Let $A = \mathscr{A}(\bar{\mathbb{D}})$. Is it true that **(a)** for each unitary $u \in A$ we have $\sigma(u) \subset \mathbb{T}$? **(b)** for each selfadjoint $a \in A$ we have $\sigma(a) \subset \mathbb{R}$? **(c)** for each $a \in A$ we have $\|a\| = r(a)$?

2.11. Let $A = \ell^1(\mathbb{Z})$. Is it true that **(a)** for each unitary $u \in A$ we have $\sigma(u) \subset \mathbb{T}$? **(b)** for each selfadjoint $a \in A$ we have $\sigma(a) \subset \mathbb{R}$? **(c)** for each $a \in A$ we have $\|a\| = r(a)$?

2.12. Let A be a unital C^*-algebra, and let $u \in A$ be a unitary element.
(a) Prove that if $\sigma(u) \neq \mathbb{T}$, then there exists a selfadjoint element $a \in A$ such that $u = \exp(ia)$.
(b) Does (a) hold if $\sigma(u) = \mathbb{T}$?

2.13. Let $\varphi \colon A \to B$ be a surjective $*$-homomorphism of C^*-algebras. Is it true that
(a) for each selfadjoint $b \in B$ there exists a selfadjoint $a \in A$ with $\varphi(a) = b$?
(b) for each unitary $b \in B$ there exists a unitary $a \in A$ with $\varphi(a) = b$?

2.14. Construct a unital Banach $*$-algebra and a normal element $a \in A$ which does not have a continuous functional calculus (i.e., there is no continuous unital $*$-homomorphism $\gamma\colon C(\sigma(a)) \to A$ satisfying $\gamma(t) = a$, where t is the tautological embedding of $\sigma(a)$ into $\mathbb{C}$).

2.15. Let X be a compact Hausdorff topological space. Show that for each $a \in C(X)$ and each $f \in C(\sigma(a))$ we have $f(a) = f \circ a$.

2.16. Let $\alpha \in \ell^\infty$, and let M_α denote the respective diagonal operator on ℓ^2. Show that for each $f \in C(\sigma(M_\alpha))$ we have $f(M_\alpha) = M_{f\circ\alpha}$.

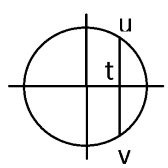
2.17. Let (X, μ) be a σ-finite measure space, let $\varphi\colon X \to \mathbb{C}$ be an essentially bounded measurable function, and let M_φ denote the respective multiplication operator on $L^2(X, \mu)$. Show that for each $f \in C(\sigma(M_\varphi))$ we have $f(M_\varphi) = M_{f\circ\varphi}$ (in particular, give a precise meaning to the expression $f \circ \varphi$).

2.18. Let A and B be unital C^*-algebras, and let $\varphi\colon A \to B$ be a unital $*$-homomorphism. Show that for each normal $a \in A$ and each $f \in C(\sigma(a))$ we have $\varphi(f(a)) = f(\varphi(a))$.

2.19. Let A be a C^*-algebra. Given a normal element $a \in A$ and $f \in C(\sigma_{A_+}(a))$, let $f(a)$ denote the result of applying to f the functional calculus for a in A_+. Show that $f(a) \in A$ iff $f(0) = 0$.

2.20. Let H be a Hilbert space. Show that
(a) $T \in \mathscr{B}(H)$ is selfadjoint $\iff$ $\langle Tx \,|\, x\rangle \in \mathbb{R}$ for all $x \in H$;
(b) $T \in \mathscr{B}(H)$ is positive (as an element of $\mathscr{B}(H)$) $\iff$ $\langle Tx \,|\, x\rangle \geqslant 0$ for all $x \in H$;
(c) $P \in \mathscr{B}(H)$ is an orthogonal projection onto a closed subspace of H $\iff$ $P = P^* = P^2$;
(d) $T \in \mathscr{B}(H)$ is an isometry (i.e., $\|Tx\| = \|x\|$ for all $x \in H$) $\iff$ $\langle Tx \,|\, Ty\rangle = \langle x \,|\, y\rangle$ for all $x, y \in H$ $\iff$ $T^*T = 1$;
(e) $U \in \mathscr{B}(H)$ is a bijective isometry $\iff$ U is a unitary element of $\mathscr{B}(H)$ (i.e., $U^*U = UU^* = 1$);
(f) give a geometric interpretation of the property $TT^* = 1$ for $T \in \mathscr{B}(H)$.

2.21. Show that each element of a unital C^*-algebra is a linear combination of four unitaries.
Hint:



Definition 2.2. Let A be a C^*-algebra. An element $p \in A$ is a *projection* if $p = p^* = p^2$ (cf. Exercise 2.20 (c)).

2.22. Let A be a C^*-algebra, and let $u \in A$.
(a) Show that u^*u is a projection iff $uu^*u = u$. An element with the above property is called a *partial isometry.*
(b) Let H be a Hilbert space. Show that $u \in \mathscr{B}(H)$ is a partial isometry iff the restriction of u to $(\operatorname{Ker} u)^\perp$ is an isometry.

2.23 (*polar decomposition*)**.** Let H be a Hilbert space. Prove that for each $a \in \mathscr{B}(H)$ there exists a unique partial isometry $u \in \mathscr{B}(H)$ such that $a = u|a|$ and $\operatorname{Ker} u = (\operatorname{Im}|a|)^\perp$.

2.24 (*polar decomposition for invertibles*)**.** Let A be a unital C^*-algebra.
(a) Show that for each invertible $a \in A$ there exists a unique unitary $u \in A$ such that $a = u|a|$.
(b) Does (a) hold if a is not invertible and $A = C[a, b]$?
(c) Does (a) hold if a is not invertible and $A = \mathscr{B}(H)$?

2.25. Let A be a C^*-algebra, and let $a, b \in A$, $0 \leqslant a \leqslant b$.
(a) Prove that $a^{1/2} \leqslant b^{1/2}$.
(b) Give an example showing that, in general, $a^2 \nleqslant b^2$.