Commutative Banach algebras

(EXERCISES FOR LECTURES 1–3)

1.1. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} = \partial\mathbb{D}$, and let $\mathscr{P}(\mathbb{T})$ denote the closure of $\mathbb{C}[z]$ in $C(\mathbb{T})$, where z is the coordinate on $\mathbb{C}$. Recall that the *disk algebra* $\mathscr{A}(\bar{\mathbb{D}})$ consists of those $f \in C(\bar{\mathbb{D}})$ that are holomorphic on $\mathbb{D}$. Show that each $f \in \mathscr{P}(\mathbb{T})$ uniquely extends to $\tilde{f} \in \mathscr{A}(\bar{\mathbb{D}})$, and that $\sigma_{\mathscr{P}(\mathbb{T})}(f) = \tilde{f}(\bar{\mathbb{D}})$.

1.2. A commutative unital algebra A is *local* if A has a unique maximal ideal. Construct a local Banach algebra $A \neq \mathbb{C}$ without zero divisors. Describe explicitly the Gelfand transform of A.

Hint. Consider the subalgebra of $\mathbb{C}[[z]]$ that consists of formal series $a = \sum c_n z^n$ satisfying $\|a\| = \sum |c_n| w_n < \infty$. Here (w_n) is a sequence of positive numbers satisfying some special conditions.

1.3. Let $V\colon L^2[0,1] \to L^2[0,1]$ denote the operator given by

$$(Vf)(x) = \int_0^x f(t)\,dt \qquad (f \in L^2[0,1]).$$

Show that the unital Banach subalgebra of $\mathscr{B}(L^2[0,1])$ generated by V (i.e., the smallest closed subalgebra of $\mathscr{B}(L^2[0,1])$ containing V and the identity operator) is local.

1.4. Let $\mathscr{O}(\mathbb{C})$ be the algebra of holomorphic functions on $\mathbb{C}$ equipped with the norm $\|f\| = \sup_{|z|\leqslant 1} |f(z)|$.

(a) Is $\mathscr{O}(\mathbb{C})$ a Banach algebra?

(b) Show that $\mathscr{O}(\mathbb{C})$ has a dense maximal ideal of infinite codimension.

1.5. **(a)** Let A be a Banach algebra, $a, b \in A$, $ab = ba$. Prove that $r(a+b) \leqslant r(a) + r(b)$ and $r(ab) \leqslant r(a)r(b)$ (where r is the spectral radius).

(b) Does (a) hold if we drop the assumption that $ab = ba$?

1.6. Prove that every proper modular ideal of an algebra A is contained in a maximal modular ideal. (*Hint:* modify the proof of the respective result for unital algebras, see the lectures.)

1.7. Let $c_{00} \subset c_0$ denote the ideal of finite sequences (i.e., of those sequences $a = (a_n)$ such that $a_n = 0$ for all but finitely many $n \in \mathbb{N}$). Prove that c_{00} is not contained in a maximal ideal of c_0.

1.8. Let $A = \{f \in C[0,1] : f(0) = 0\}$, and let $I = \{f \in A : f \text{ vanishes on a neighborhood of } 0\}$. Prove that I is not contained in a maximal ideal of A.

1.9. Construct a commutative Banach algebra which has a dense proper ideal. (Clearly, A cannot be unital, see the lectures.)

1.10. Let X be a compact Hausdorff topological space. For each closed subset $Y \subset X$ let $I_Y = \{f \in C(X) : f|_Y = 0\}$. Prove that the assignment $Y \mapsto I_Y$ is a 1-1 correspondence between the collection of all closed subsets of X and the collection of all closed ideals of $C(X)$.

1.11. A commutative algebra A is *semisimple* if the intersection of all maximal modular ideals of A (the *Jacobson radical* of A) is $\{0\}$. Show that every homomorphism from a Banach algebra to a commutative semisimple Banach algebra is continuous.

1.12. Describe the maximal spectrum and the Gelfand transform for the algebras **(a)** $C^n[0,1]$; **(b)** $\mathscr{A}(\bar{\mathbb{D}})$; **(c)** $\mathscr{P}(\mathbb{T})$; **(d)** $\ell^1(\mathbb{Z})$.

1.13. Let $A(\mathbb{T}) = \{f \in C(\mathbb{T}) : \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty\}$, where $\hat{f}(n)$ is the nth Fourier coefficient of f w.r.t. the trigonometric system (e_n) on $\mathbb{T}$ (i.e., $e_n(z) = z^n$ for all $z \in \mathbb{T}$ and $n \in \mathbb{Z}$). Prove that $A(\mathbb{T})$ is a spectrally invariant subalgebra of $C(\mathbb{T})$.

1.14. Let X be a topological space, let $\beta X = \operatorname{Max} C_b(X)$, and let $\varepsilon\colon X \to \beta X$ take each $x \in X$ to the evaluation map $\varepsilon_x\colon C_b(X) \to \mathbb{C}$ given by $\varepsilon_x(f) = f(x)$.
(a) Prove that $(\beta X, \varepsilon)$ is the Stone-Čech compactification of X (i.e., for each compact Hausdorff topological space and each continuous map $f\colon X \to Y$ there exists a unique continuous map $\tilde{f}\colon \beta X \to Y$ such that $\tilde{f} \circ \varepsilon = f$).
(b) Prove that $\varepsilon(X)$ is dense in βX.
(c) Prove that ε is a homeomorphism onto $\varepsilon(X)$ if and only if X is completely regular.

1.15. Let G be a discrete group, and let $\widehat{G}$ be the *Pontryagin dual* of G, i.e., the group of all homomorphisms from G to $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The standard topology on $\widehat{G}$ is the topology of pointwise convergence (i.e., the topology inherited from $\mathbb{T}^G$).
(a) Construct a homeomorphism $\operatorname{Max}(\ell^1(G)) \cong \widehat{G}$.
(b) Show that, under the identification $\operatorname{Max}(\ell^1(G)) \cong \widehat{G}$, the Gelfand transform of $\ell^1(G)$ becomes the *Fourier transform*

$$\mathscr{F}: \ell^1(G) \to C(\widehat{G}), \quad (\mathscr{F}f)(\chi) = \sum_{x\in G} f(x)\chi(x).$$

1.16. Let A be a commutative algebra, and I be a maximal ideal of A. Prove that I is either modular or a codimension 1 ideal containing $A^2 = \operatorname{span}\{ab : a, b \in A\}$. As a consequence, if $A^2 = A$, then all maximal ideals of A are modular.

1.17. Consider the Banach algebra $\ell^2 = \ell^2(\mathbb{N})$ with pointwise multiplication. Show that ℓ^2 has maximal ideals which are not modular.

1.18. Let A be a commutative algebra, and let $\operatorname{Max}_+(A) = \operatorname{Max}(A) \cup \{A\}$. Prove that the map $\operatorname{Max}(A_+) \to \operatorname{Max}_+(A)$, $I \mapsto I \cap A$, is a bijection.

1.19. Let A be a commutative Banach algebra, and let I be a closed ideal of A.
(a) Construct a homeomorphism between $\operatorname{Max}(A/I)$ and a closed subset of $\operatorname{Max}(A)$.
(b) Show that each nonzero character $I \to \mathbb{C}$ uniquely extends to a character $A \to \mathbb{C}$. Show that the resulting map $\operatorname{Max}(I) \to \operatorname{Max}(A)$ is a homeomorphism onto an open subset of $\operatorname{Max}(A)$.

1.20. Let A be a commutative Banach algebra. Show that the Gelfand transform $\Gamma\colon A \to C_0(\operatorname{Max} A)$ is a topological embedding if and only if there exists $c > 0$ such that $\|a^2\| \geqslant c\|a\|^2$ for all $a \in A$.

1.21. Construct a commutative Banach algebra A such that for each $t \in [0,1]$ there exists a character χ of A with $\|\chi\| = t$. (Clearly, A cannot be unital, see the lectures.)

1.22. Let A and B be commutative unital Banach algebras, and let $\varphi\colon A \to B$ be a continuous unital homomorphism.
(a) Show that, if $\overline{\varphi(A)} = B$, then $\varphi^*\colon \operatorname{Max}(B) \to \operatorname{Max}(A)$ is a topological embedding.
(b) Suppose that φ^* is a homeomorphism. Does this imply that $\overline{\varphi(A)} = B$?
(c) Suppose that φ^* is a homeomorphism. Does this imply that φ is injective?